Supplemental Materials: Interpoint-Ranking Sign Covariance

In this supplementary material we prove technical results and present additional simulation results in the article “Interpoint-Ranking Sign Covariance” by Moon and Chen. All references, as well as theorem and equation numbering, refer to the original paper.

S-I. Proofs

LEMMA 1. *Assume θ is discrete or continuous, or a mixture of the two, that is, assume there exists a probability mass function P_{XY} and a density function h such that*

$$\theta(A \times B) = \sum_{x_i, y_i} P_{XY}(x_i, y_i) + \int_{A\times B} h(x,y) G(dx)G(dy),$$

where $A \subset \mathcal{X}, B \subset \mathcal{Y}$ are any two open sets and G is the abstract Wiener measure on $\mathcal{X}$ and $\mathcal{Y}$. Then, induced random variables defined by $U_x = \rho(x, X)$, $V_y = \zeta(y, Y)$ with $U_x : \mathcal{X} \to \mathbb{R}$, $V_y : \mathcal{Y} \to \mathbb{R}$ are well-defined and has a jointly discrete or continuous distribution, or a mixture of the two.

Proof. Denote the closed ball with center x and radius r_1 in $\mathcal{X}$ as $\bar{B}_\rho(x, r_1)$ or $\bar{B}(x, r_1)$, and the closed ball with center y and radius r_2 in $\mathcal{Y}$ as $\bar{B}_\zeta(y, r_2)$ or $\bar{B}(y, r_2)$.

For U_x,

$$pr(U_x \in [0, a]) = pr(X \in \bar{B}(x, a)) = \mu(\bar{B}(x, a)).$$

So U_x is a well-defined Borel probability measure on $\mathbb{R}$. Same applied for V_y and the joint variable.

Now we show that U_x and V_y has a jointly discrete, continuous or a mixture of two distributions.

$$\begin{aligned} pr(U_x \leq a, V_y \leq b) &= pr(\bar{B}(x,a) \times \bar{B}(y,b)) \\ &= \sum_{a' \in [0,a], b' \in [0,b]} \sum_{x_i \in \partial B(x,a'), y_i \in \partial B(y,b')} P_{XY}(x_i, y_i) \\ &\quad + \int_0^a \int_0^b \int_{\partial B(x,a') \times \partial B(y,b')} h(x,y) G(dx) G(dy) db' da' \end{aligned}$$

Here, $P_{U_x V_y}(a', b') = \sum_{x_i \in \partial \bar{B}(x,a'), y_i \in \partial \bar{B}(y,b')} P_{X,Y}(x_i, y_i)$ is a probability mass function and $\int_{\partial B(x,a') \times \partial B(y,b')} h(x,y) G(dx) G(dy)$ is a density function. □

Proof of Theorem 1

Proof. Let’s define $\eta(x, y)$ for $(x, y) \in \mathcal{X} \times \mathcal{Y}$ as

$$\eta(x,y) = E[a\{\rho(x,X^1), \rho(x,X^2), \rho(x,X^3), \rho(x,X^4)\} a\{\zeta(y,Y^1), \zeta(y,Y^2), \zeta(y,Y^3), \zeta(y,Y^4)\}],$$

where $(X^1, Y^1), \ldots, (X^4, Y^4)$ are independent copies of (X, Y). If we define new random variables induced by x, y as $U_x = \rho(x, X)$, $V_y = \zeta(y, Y)$, respectively, which are well-defined by Lemma 1, $\eta(x, y)$ becomes $\tau^*(U_x, V_y)$. Therefore, under the consistency condition of τ^* met by Lemma 1, $\eta(x, y)$ is zero if U_x and V_y are independent and positive otherwise. Then IPR-$\tau^*(X, Y) = E_{(x,y)\sim\theta}\{\eta(x, y)\} \geq 0$ is derived.

If X and Y are independent, then U_x and V_y are independent for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$. So IPR-$\tau^*(X, Y) = 0$. In the following, we will show that IPR-$\tau^*(X, Y) = 0$ only if X and Y are independent.

Step 1) We claim that if $\theta \neq \mu \times \nu$, there exist a point of dependence in the support set of θ. Since $h(x, y)$ is continuous on any continuous point of θ, we have marginal density funtion $f(x)$ and $g(y)$ for any continuous point (x, y) of θ. Denote support sets of θ, μ and ν as S_θ, S_μ and S_ν.

Then, since $S_\theta \subset S_\mu \times S_\nu$, we have

$$
\begin{aligned}
1 &= \sum_{S_\mu \times S_\nu} P_{X,Y}(x, y) + \int_{S_\mu \times S_\nu} h(x, y) G(dx \times dy) \\
&= \sum_{S_\theta} P_X(x) P_Y(y) + \sum_{S_\mu \times S_\nu / S_\theta} P_X(x) P_Y(y) \\
&\quad + \int_{S_\theta} f(x) g(y) G(dx \times dy) + \int_{S_\mu \times S_\nu / S_\theta} f(x) g(y) G(dx \times dy).
\end{aligned}
$$

If the claim is not true, we must either have a discrete point $(x, y) \in S_\theta^c$ such that $P_{X,Y}(x, y) \neq P_X(x) P_Y(y)$, and/or sets A and B such that $\int_{A \times B} h(x, y) G(dx \times dy) \neq \int_A f(x) G(dx) \int_B g(x) G(dy)$. In the first case, $P_{X,Y}(x, y) \neq P_X(x) P_Y(y)$ implies $P_X(x) P_Y(y) > 0$, so $(x, y) \in S_\mu \times S_\nu / S_\theta$. Then there should exists another point $(x', y') \in S_\theta$ such that $P_{X,Y}(x', y') > P_X(x') P_Y(y')$ to balance the above equation. The same argument applies to the later case.

Step 2) With the results of *Step 1)*, if (x, y) is a discrete point and $P_{X,Y}(x, y) \neq P_X(x) P_Y(y)$, we can find balls $B_\rho(x, r_1)$ and $B_\zeta(y, r_2)$ such that

$$
\theta(B_\rho(x, r_1) \times B_\zeta(y, r_2)) \neq \mu(B_\rho(x, r_1)) \nu(B_\zeta(y, r_2)),
$$

with small enough r_1 and r_2. Since $\{U_x < r_1\} = B_\rho(x, r_1)$ and $\{V_y < r_2\} = B_\zeta(y, r_y)$, this equation is reduced to

$$
P_{U_x, V_y}(r_1, r_2) \neq P_{U_x}(r_1) P_{V_y}(r_2).
$$

So U_x and V_y are not independent, i.e. $\eta(x, y) > 0$. Then,

$$
\text{IPR-}\tau^*(X, Y) \geq \eta(x, y) \theta(x, y) > 0.
$$

If (x, y) is a continuous point, we can say that $h(x, y) > f(x) g(y)$ without a loss of generosity. Since $h(x, y)$ is continuous, so are $f(x)$, $g(y)$. We can find an area A of nonzero measure such that there exist balls $B_\rho(v, r_v)$, $r_v > 0$ and $B_\zeta(w, r_w)$, $r_w > 0$ for every $(v, w) \in A$ where $h(v', w') > f(v') g(w')$ for $v' \in B_\rho(v, r_v)$ and $w' \in B_\zeta(w, r_w)$.

Then,

$$
\begin{aligned}
\theta(B_\rho(v, r_v) \times B_\zeta(w, r_w)) &= \int_{B_\rho(v, r_v) \times B_\zeta(w, r_w)} h(v', w') G(dv' \times dw') \\
&> \int_{B_\rho(v, r_v)} f(w) G(dw) \int_{B_\zeta(w, r_w)} g(w') G(dw') \\
&= \mu(B_\rho(v, r_v)) \nu(B_\zeta(w, r_w))
\end{aligned}
$$

Same as the discrete case, the inequality is reduced to $P_{U_v, V_y}((-\infty, r_v) \times (-\infty, r_w)) > P_{U_v}((-\infty, r_v)) P_{V_w}((-\infty, r_w))$. So U_v and V_w are not independent and $\eta(v, w) > 0$ for every

$(v, w) \in A$. Then,

$$\text{IPR-}\tau^*(X, Y) \geq \int_A \eta(v, w) h(v, w) G(dv \times dw) > 0,$$

and IPR-$\tau^*(X, Y) = 0$ only if X and Y are independent.

S-II. RESULTS FOR SIMULATION III

In Simulation III, we consider variables in a non-Euclidean space with Riemannian Metric as a distance. We first consider variables on a spherical coordinate of the unit sphere S^2. In a spherical coordination, each point on a sphere, denoted by $(\theta, \phi)_s$, is uniquely represented with longitude θ and latitude ϕ, where θ specifies the east-west position on a spherical surface and ϕ specifies an angle which range from 0 at the equator to $\pi/2$ at the north, and $-\pi/2$ at the south; so $\theta \in [-\pi, \pi]$ and $\phi \in [-\pi/2, \pi/2]$. We use the great-circle distance, the shortest distance over the surface of a sphere between two points.

*Example S*1. (Type I) $X = (X_1, X_2)$ where $X_1, X_2 \sim \mathcal{U}(0, 1)$, $Y = (\theta, \phi)_s \in S^2$ with $\theta \sim \mathcal{U}(-\pi, \pi)$, $\phi \sim \mathcal{U}(-\pi/2, \pi/2)$.

*Example S*2. (Sphere 1) X is same as Example $S1$, $Y = (\theta, \phi)_s \in S^2$ with $\theta \sim \mathcal{U}(-\pi, \pi)$, $\phi = \pi(X_1 + X_2)\epsilon/2 - \pi/2$, $\epsilon \sim \mathcal{U}(0, 1)$.

*Example S*3. (Sphere 2) X is same as Example $S1$, $Y = (\theta, \phi)_s \in S^2$ with $\theta \sim \mathcal{U}(-\pi, \pi)$, $\phi = \pi|X_1 - X_2|\epsilon - \pi/2$, $\epsilon \sim \mathcal{U}(0, 1)$.

*Example S*4. (Sphere 3) X is same as Example $S1$, $Y = (\theta, \phi)_s \in S^2$ with $\theta \sim \mathcal{U}(-\pi, \pi)$, $\phi = \pi(X_1 + X_2)^2\epsilon/4 - \pi/2$, $\epsilon \sim \mathcal{U}(0, 1)$.

We also consider symmetric positive matrices. Specifically, we consider a 3 by 3 symmetric positive matrix variable whose every non-diagonal element equal to ρ. We use the affine invariant Riemannian metric, $d(A, B) = ||log(A^{-1/2}BA^{-1/2})||_F$, where $log(A)$ is the matrix logarithm of A, and $||A||_F$ is the Frobenius norm of A.

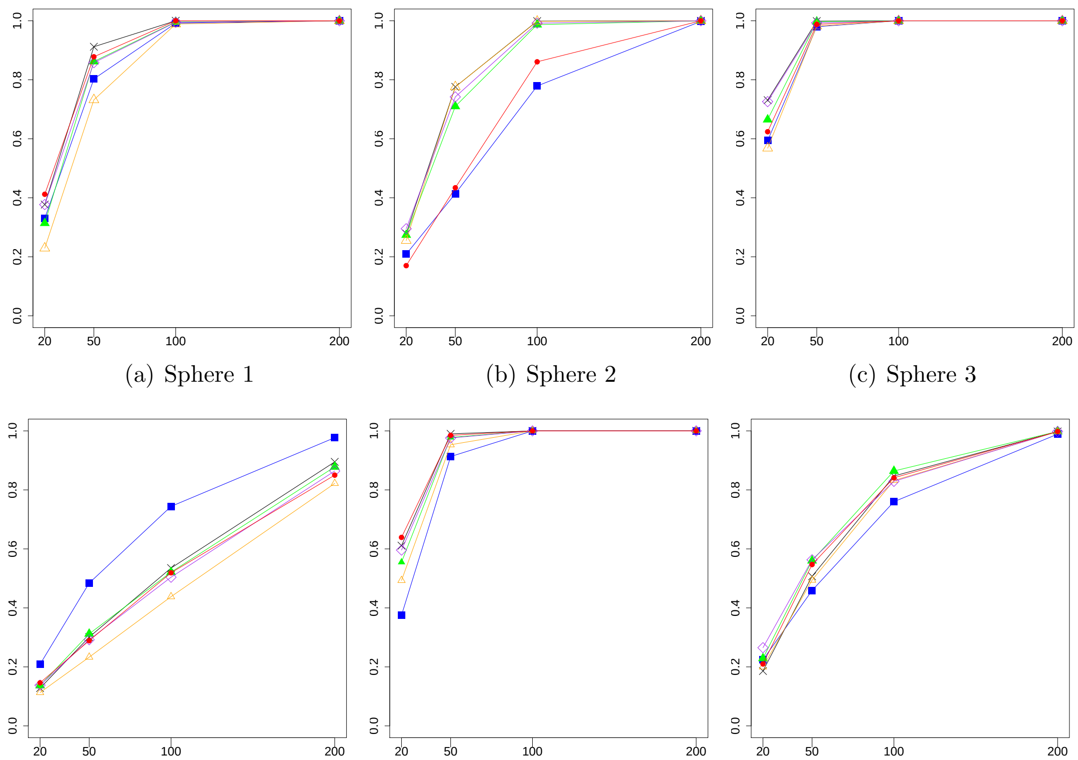
*Example S*5. (Type I) $X = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$, $Y = \epsilon$ with $\rho \sim \mathcal{U}(0, 0.3)$, $\epsilon \sim N(0, 0.3)$.

*Example S*6. (PD 1) $X = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$, $Y = \rho + \epsilon$ with $\rho \sim \mathcal{U}(0, 0.3)$, $\epsilon \sim N(0, 0.3)$.

*Example S*7. (PD 2) $X = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$, $Y = \epsilon$ with $\rho \sim \mathcal{U}(0, 0.3)$, $\epsilon \sim N(0, \rho/3)$.

*Example S*8. (PD 3) $X = \begin{pmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_1 \\ \rho_1 & \rho_1 & 1 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & \rho_2 & \rho_2 \\ \rho_2 & 1 & \rho_2 \\ \rho_2 & \rho_2 & 1 \end{pmatrix}$, $\rho_1 = 1/(1 + \lambda_1^2)$, $\rho_2 = 1/(1 + \lambda_2^2)$ with $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \sim N(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix})$.

We observed that the Type-I error rates for all methods are well-controlled. The empirical powers are shown in Figure S1. All methods have increasing power towards 1 as the sample size increases. The "dCov" method seems to perform the best in linear relationships and is less competitive in other settings.



(a) Sphere 1 (b) Sphere 2 (c) Sphere 3

(d) PD 1 (e) PD 2 (f) PD 3

Fig. S1. Simulation III: Empirical power of the tests for IPR-τ^* (●, red), dCov (■, blue), HHG (×, black), BCov1 (▲, green), BCov2 (△, orange) and HISC (◇, purple). Power values are computed for each of the sample sizes 20, 50, 100, 200 with 1,000 simulations.