Biometrika (-), **-**, -, *pp*. 1–6
Printed in Great Britain -

Supplementary material for A test of weak separability for multi-way functional data, with application to brain connectivity studies - Proofs

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Proof of Lemma 1. Let $f_j \quad (j = 1, 2, \ldots)$ and $g_k \quad (k = 1, 2, \ldots)$ be a pair of bases that satisfies weak separability. For $(j, k) \neq (j', k')$, we have $\langle Cf_j \otimes g_k, f_{j'} \otimes g_{k'} \rangle = E\left(\langle X - \mu, f_j \otimes g_k \rangle \langle X - \mu, f_{j'} \otimes g_{k'} \rangle\right) = 0$. Since the covariance operator C is diagonalized under the orthonormal basis $f_j \otimes g_k$ $(j = 1, 2, \ldots;\ k = 1, 2, \ldots)$, by Mercer's theorem,

$$C(s, t; u, v) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \eta_{jk} f_j(s) g_k(t) f_j(u) g_k(v),$$

where $\eta_{jk} = \langle Cf_j \otimes g_k, f_j \otimes g_k \rangle = \mathrm{var}\left(\langle X - \mu, f_j \otimes g_k \rangle\right)$, and the convergence is absolute and uniform.

The marginal kernel can then be written as

$$\begin{aligned} C_{\mathcal{S}}(s, u) &= \int_{\mathcal{T}} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \eta_{jk} f_j(s) g_k(t) f_j(u) g_k(t) dt \\ &= \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \eta_{jk} \right) f_j(s) f_j(u). \end{aligned}$$

The exchange of the integral and sums is allowed by the Fubini–Tonelli theorem, by noticing that

$$\begin{aligned} &\int_{\mathcal{T}} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\eta_{jk} f_j(s) g_k(t) f_j(u) g_k(t)| dt \\ &\leq \int_{\mathcal{T}} \left\{ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \eta_{jk} f_j^2(s) g_k^2(t) \right\}^{1/2} \left\{ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \eta_{jk} f_j^2(u) g_k^2(t) \right\}^{1/2} dt \\ &= \int_{\mathcal{T}} C(s, t; s, t)^{1/2} C(u, t; u, t)^{1/2} dt \\ &\leq \int_{\mathcal{T}} \sup_{s,t} |C(s, t, s, t)| dt \leq \infty, \end{aligned}$$

where we use the Cauchy–Schwarz inequality.

Thus, we see that the f_j are eigenfunctions of $C_{\mathcal{S}}$ with eigenvalues $\lambda_j = \sum_{k=1}^{\infty} \eta_{jk}$. An analogous computation shows that the g_k are eigenfunctions of $C_{\mathcal{T}}$ with eigenvalues $\gamma_k = \sum_{j=1}^{\infty} \eta_{jk}$.

Proof of Lemma 2. With strong separability, we have $C(s,t;u,v) = aC_1(s,u)C_2(t,v)$. From the definition of $C_{\mathcal{S}}$, we have

$$C_{\mathcal{S}}(s,u) = \int_{\mathcal{T}} C(s,t;u,t)dt = aC_1(s,u)\int_{\mathcal{T}} C_2(t,t)dt = aC_1(s,u).$$

An analogous argument shows $C_{\mathcal{T}}(t,v) = aC_2(t,v)$. Note that $a = \int_{\mathcal{T}}\int_{\mathcal{S}} C(s,t;s,t)dsdt$. If we use the marginal eigenfunctions ψ_j and ϕ_k as the bases, it is easy to show that when $(j,k) \neq (j',k')$, $\operatorname{cov}(\chi_{jk}, \chi_{j'k'}) = \int_{\mathcal{T},\mathcal{S},\mathcal{T},\mathcal{S}} C(s,t;u,v)\psi_j(s)\phi_k(t)\psi_{j'}(u)\phi_{k'}(v)dsdtdudv = 0$. Thus, we have weak separability.

Proof of Lemma 3. When V is of rank 1, V can be written $V = WZ^T$, where W and Z are column vectors with entries $(w_1, w_2, \ldots)$ and $(z_1, z_2, \ldots)$, respectively. Thus, $\eta_{jk} = w_j z_k$, and under weak separability, Equation (3) in the paper can be written

$$\begin{aligned} C(s,t;u,v) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} w_j z_k \psi_j(s)\psi_j(u)\phi_k(t)\phi_k(v) \\ &= \left\{\sum_{j=1}^{\infty} w_j\psi_j(s)\psi_j(u)\right\}\left\{\sum_{k=1}^{\infty} z_k\phi_k(t)\phi_k(v)\right\}. \end{aligned}$$

The above can be normalized to fit the definition of strong separability in Lemma 2.

Under strong separability, from the proof of Lemma 2 we have

$$C(s,t;u,v) = \frac{1}{\int_{\mathcal{T}}\int_{\mathcal{S}} C(s,t;s,t)dsdt} C_{\mathcal{S}}(s,u)C_{\mathcal{T}}(t,v),$$

so $\eta_{jk} = \{1/\int_{\mathcal{T}}\int_{\mathcal{S}} C(s,t;s,t)dsdt\}\lambda_j\gamma_k$, and then $V = \{1/\int_{\mathcal{T}}\int_{\mathcal{S}} C(s,t;s,t)dsdt\}\Lambda\Gamma^T$.

Proof of Theorem 1. For H_1 and H_2 two real separable Hilbert spaces, we further define the partial trace with respect to H_1 as the unique bounded linear operator $tr_1 : \mathcal{B}_{Tr}(H_1 \otimes H_2) \to \mathcal{B}_{Tr}(H_2)$ satisfying $tr_1(C_1 \tilde{\otimes} C_2) = tr(C_1)C_2$ for all $C_1 \in \mathcal{B}_{Tr}(H_1)$, $C_2 \in \mathcal{B}_{Tr}(H_2)$. The partial trace with respect to H_2 is defined symmetrically and denoted by tr_2. With the notation of partial trace, we can see that $C_{\mathcal{T}} = tr_1(C)$ and $C_{\mathcal{S}} = tr_2(C)$. The estimated marginal covariance operators can also be written as $\hat{C}_{\mathcal{S}} = tr_2(C_n)$ and $\hat{C}_{\mathcal{T}} = tr_1(C_n)$. We use these equalities in proofs but not in computation. In practice, the estimated marginal covariances are calculated without having to calculate C_n.

We use similar notation and conditions as used by Aston et al. (2017). However, to derive the asymptotic distribution of their test statistic for strong separability, they focus on deriving the asymptotic distribution of the difference between the sample covariance operator and its strong separable approximation. Then by projecting on the estimated marginal eigenfunctions, they check the requirement for strong separability that $\eta_{jk} = a\lambda_j\gamma_k$. They do not need further results on the estimation errors of the marginal eigenfunctions and random scores besides that they are consistent. By contrast, our proofs involve the expansion of $\hat{\psi}_j - \psi_j$ and $\hat{\phi}_k - \phi_k$, and four-way tensor products with indices (j,k,j',k').

From Condition 1 in Section 3·2 and the remark following it, $\mathcal{Z}_n = n^{1/2}(C_n - C)$ converges to a Gaussian random element in $\mathcal{B}_{Tr}\{L^2(\mathcal{S}\times\mathcal{T})\}$ with mean 0 and covariance structure $\Sigma_C = \mathrm{E}[\{(X-\mu)\otimes(X-\mu) - C\}\tilde{\otimes}\{(X-\mu)\otimes(X-\mu) - C\}]$.

For T_n as defined in Equation (4) in the paper,

$$T_n(j,k,j',k') = n^{1/2}\langle C_n(\hat{\psi}_j \otimes \hat{\phi}_k), \hat{\psi}_{j'} \otimes \hat{\phi}_{k'}\rangle = n^{1/2} tr\{(\hat{\psi}_j \otimes \hat{\psi}_{j'})\tilde{\otimes}(\hat{\phi}_k \otimes \hat{\phi}_{k'})C_n\}.$$

Using (5.1.8) in Hsing & Eubank (2015), we have

$$(\hat{\psi}_j - \psi_j) = \mathcal{M}_j(\hat{C}_{\mathcal{S}} - C_{\mathcal{S}})\psi_j + o_p(\hat{\psi}_j - \psi_j),$$

where $\mathcal{M}_j = \sum_{m\neq j}(\lambda_j - \lambda_m)^{-1}\psi_m \otimes \psi_m \in \mathcal{B}_{Tr}(\mathcal{S})$ and λ_j is the jth eigenvalue of $C_{\mathcal{S}}$. Analogously,

$$(\hat{\phi}_k - \phi_k) = \mathcal{M}'_k(\hat{C}_{\mathcal{T}} - C_{\mathcal{T}})\phi_k + o_p(\hat{\phi}_k - \phi_k),$$

where $\mathcal{M}'_k = \sum_{m\neq k}(\gamma_k - \gamma_m)^{-1}\phi_m \otimes \phi_m \in \mathcal{B}_{Tr}(\mathcal{T})$ and γ_k is the kth eigenvalue of $C_{\mathcal{T}}$. Here, Condition 2 is used to guarantee that $\mathcal{M}_j$ and $\mathcal{M}'_k$ exist for $j = 1,\dots,P$ and $k = 1,\dots,K$.

Using $\hat{C}_{\mathcal{S}} - C_{\mathcal{S}} = tr_2(C_n - C)$ and $\hat{C}_{\mathcal{T}} - C_{\mathcal{T}} = tr_1(C_n - C)$, we can write $T_n(j,k,j',k')$ as

$$\begin{aligned}
T_n(j,k,j',k') =& n^{1/2} tr\left\{(\psi_j \otimes \psi_{j'})\tilde{\otimes}(\phi_k \otimes \phi_{k'})C\right\} \qquad (1)\\
&+ n^{1/2} tr\left\{(\psi_j \otimes \psi_{j'})\tilde{\otimes}(\phi_k \otimes \phi_{k'})(C_n - C)\right\}\\
&+ n^{1/2} tr\left((\psi_j \otimes \psi_{j'})\tilde{\otimes}[\phi_k \otimes \{\mathcal{M}'_{k'} tr_1(C_n - C)\phi_{k'}\}]C\right)\\
&+ n^{1/2} tr\left((\psi_j \otimes \psi_{j'})\tilde{\otimes}[\{\mathcal{M}'_k tr_1(C_n - C)\phi_k\} \otimes \phi_{k'}]C\right)\\
&+ n^{1/2} tr\left([\psi_j \otimes \{\mathcal{M}_{j'} tr_2(C_n - C)\psi_{j'}\}]\tilde{\otimes}(\phi_k \otimes \phi_{k'})C\right)\\
&+ n^{1/2} tr\left([\{\mathcal{M}_j tr_2(C_n - C)\psi_j\} \otimes \psi_{j'}]\tilde{\otimes}(\phi_k \otimes \phi_{k'})C\right)\\
&+ o_p(1).
\end{aligned}$$

The first term in the above equation is zero under H_0, since under H_0 we have the representation $C(s,t,u,v) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\eta_{jk}\psi_j(s)\psi_j(u)\phi_k(t)\phi_k(v)$, where $\eta_{jk} = \mathrm{var}(\chi_{jk})$. Also, by Proposition C·1 in Aston et al. (2017), we have that $tr\{A tr_1(T)\} = tr\{(Id_1\tilde{\otimes}A)T\}$, where Id_1 is an identity operator on $\mathcal{S}$, $A \in \mathcal{B}(\mathcal{T})$, and $T \in \mathcal{B}_{Tr}(\mathcal{S}\times\mathcal{T})$. An analogous identity holds for $tr_2(T)$. Using these facts, we give a simplified form of $T_n(j,k,j',k')$ under H_0 for 3 cases:
(Case i) $j \neq j'$ and $k \neq k'$:

$$T_n(j,k,j',k') = tr\left[\{(\psi_j \otimes \psi_{j'})\tilde{\otimes}(\phi_k \otimes \phi_{k'})\}\mathcal{Z}_n\right] + o_p(1).$$

(Case ii) $j = j'$ and $k \neq k'$:

$$\begin{aligned}
T_n(j,k,j',k') =& tr\left[\{(\psi_j \otimes \psi_{j'})\tilde{\otimes}(\phi_k \otimes \phi_{k'})\}\mathcal{Z}_n\right]\\
&+ tr\left([Id_1\tilde{\otimes}\{\eta_{jk'}(\phi_k \otimes \phi_{k'})\mathcal{M}'_k\}]\mathcal{Z}_n\right)\\
&+ tr\left([Id_1\tilde{\otimes}\{\eta_{jk}(\phi_{k'} \otimes \phi_k)\mathcal{M}'_{k'}\}]\mathcal{Z}_n\right) + o_p(1).
\end{aligned}$$

(Case iii) $j \neq j'$ and $k = k'$:

$$\begin{aligned}
T_n(j,k,j',k') =& tr\left[\{(\psi_j \otimes \psi_{j'})\tilde{\otimes}(\phi_k \otimes \phi_{k'})\}\mathcal{Z}_n\right]\\
&+ tr\left([\{\eta_{jk}(\psi_{j'} \otimes \psi_j)\mathcal{M}_{j'}\}\tilde{\otimes}Id_2]\mathcal{Z}_n\right)\\
&+ tr\left([\{\eta_{j'k}(\psi_j \otimes \psi_{j'})\mathcal{M}_j\}\tilde{\otimes}Id_2]\mathcal{Z}_n\right) + o_p(1).
\end{aligned}$$

In each of the above cases, two or more of the terms in Equation (1) end up being zero due to the orthogonality of the eigenfunctions. The latter 2 cases can be simplified to get the result in the statement of the theorem by noting that $\eta_{jk'}(\phi_k \otimes \phi_{k'})\mathcal{M}'_k = \eta_{jk'}(\gamma_k - \gamma_{k'})^{-1}\phi_k \otimes \phi_{k'}$ and $\eta_{jk}(\psi_{j'} \otimes \psi_j)\mathcal{M}_{j'} = \eta_{jk}(\lambda_{j'} - \lambda_j)^{-1}\psi_{j'} \otimes \psi_j$.

Proof of Corollary 1. From Theorem 1, we can see that all the terms of $T_n(j,k,j',k')$ can be written in the form $tr\{(A_1 \tilde{\otimes} A_2)\mathcal{Z}_n\}$ for some $A_1 \in \mathcal{B}(\mathcal{S})$ and $A_2 \in \mathcal{B}(\mathcal{T})$. Since $\mathcal{Z}_n$ converges to a Gaussian random element and $tr\{(A_1 \tilde{\otimes} A_2)\mathcal{Z}_n\}$ is a continuous linear mapping, the $T_n(j,k,j',k')$ are asymptotically jointly Gaussian for different sets of (j,k,j',k'). Let Θ be the covariance structure of the asymptotic joint distribution of the $T_n(j,k,j',k')$, and define $\mathcal{Z}$ to be a Gaussian random element with the limiting distribution of $\mathcal{Z}_n$. By the continuous mapping theorem, Θ can be calculated from terms of the form

$$\mathrm{E}[tr\{(A_1 \tilde{\otimes} A_2)\mathcal{Z}\}tr\{(B_1 \tilde{\otimes} B_2)\mathcal{Z}\}] = tr\left\{(A_1 \tilde{\otimes} A_2)\widetilde{\bigotimes}(B_1 \tilde{\otimes} B_2)\Sigma_C\right\}, \tag{2}$$

where Σ_C is defined as in the proof of Theorem 1.

Recall the Karhunen–Loève expansion of the process $X(s,t) = \mu(s,t) + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\chi_{jk}\psi_j(s)\phi_k(t)$. We define $u_{ij} = \psi_i \otimes \psi_j \in \mathcal{B}_{HS}(\mathcal{S})$, $v_{ij} = \phi_i \otimes \phi_j \in \mathcal{B}_{HS}(\mathcal{T})$, $\beta_{i,i',j,j',k,k',l,l'} = \mathrm{E}(\chi_{ii'}\chi_{jj'}\chi_{kk'}\chi_{ll'})$, and $\eta_{ii'} = \mathrm{E}(\chi_{ii'}^2)$. With weak separability, we have

$$\begin{aligned} & tr\left\{(A_1 \tilde{\otimes} A_2)\widetilde{\bigotimes}(B_1 \tilde{\otimes} B_2)\Sigma_C\right\} \\ &= \sum_{i,i',j,j',k,k',l,l'} \beta_{i,i',j,j',k,k',l,l'} tr(A_1 u_{ij}) tr(A_2 v_{i'j'}) tr(B_1 u_{kl}) tr(B_2 v_{k'l'}) \\ &\quad - \sum_{i,i',j,j'} \eta_{ii'}\eta_{jj'} tr(A_1 u_{ii}) tr(B_1 u_{jj}) tr(A_2 v_{i'i'}) tr(B_2 v_{j'j'}). \end{aligned}$$

Each of the trace terms in the above equation can be evaluated using the identities $tr(Id_1 u_{ij}) = I(i=j)$, $tr(Id_2 v_{i'j'}) = I(i'=j')$, $tr\{(\psi_{j_1} \otimes \psi_{j_1'})u_{ij}\} = I(i=j_1)I(j=j_1')$, and $tr\{(\phi_{k_1} \otimes \phi_{k_1'})v_{i'j'}\} = I(i'=k_1)I(j'=k_1')$. From these identities and the possible forms of A_1, A_2, B_1, and B_2 given in Theorem 1, it follows that the second sum is always 0. The first sum can be simplified by considering 9 cases, as follows:

(Case 1) $A_1 = a_1\psi_{j_1} \otimes \psi_{j_1'}$, $A_2 = a_2\phi_{k_1} \otimes \phi_{k_1'}$, $B_1 = b_1\psi_{j_2} \otimes \psi_{j_2'}$, $B_2 = b_2\phi_{k_2} \otimes \phi_{k_2'}$:

$$tr\left\{(A_1 \tilde{\otimes} A_2)\widetilde{\bigotimes}(B_1 \tilde{\otimes} B_2)\Sigma_C\right\} = a_1a_2b_1b_2\beta_{j_1,k_1,j_1',k_1',j_2,k_2,j_2',k_2'}.$$

(Case 2) $A_1 = Id_1$, $A_2 = a_2\phi_{k_1} \otimes \phi_{k_1'}$, $B_1 = b_1\psi_{j_2} \otimes \psi_{j_2'}$, $B_2 = b_2\phi_{k_2} \otimes \phi_{k_2'}$:

$$tr\left\{(A_1 \tilde{\otimes} A_2)\widetilde{\bigotimes}(B_1 \tilde{\otimes} B_2)\Sigma_C\right\} = a_2b_1b_2\sum_{i=1}^{\infty}\beta_{i,k_1,i,k_1',j_2,k_2,j_2',k_2'}.$$

(Case 3) $A_1 = a_1\psi_{j_1} \otimes \psi_{j_1'}$, $A_2 = Id_2$, $B_1 = b_1\psi_{j_2} \otimes \psi_{j_2'}$, $B_2 = b_2\phi_{k_2} \otimes \phi_{k_2'}$:

$$tr\left\{(A_1 \tilde{\otimes} A_2)\widetilde{\bigotimes}(B_1 \tilde{\otimes} B_2)\Sigma_C\right\} = a_1b_1b_2\sum_{i'=1}^{\infty}\beta_{j_1,i',j_1',i',j_2,k_2,j_2',k_2'}.$$

(Case 4) $A_1 = a_1\psi_{j_1} \otimes \psi_{j_1'}$, $A_2 = a_2\phi_{k_1} \otimes \phi_{k_1'}$, $B_1 = Id_1$, $B_2 = b_2\phi_{k_2} \otimes \phi_{k_2'}$:

$$tr\left\{(A_1 \tilde{\otimes} A_2)\widetilde{\bigotimes}(B_1 \tilde{\otimes} B_2)\Sigma_C\right\} = a_1a_2b_2\sum_{k=1}^{\infty}\beta_{j_1,k_1,j_1',k_1',k,k_2,k,k_2'}.$$

(Case 5) $A_1 = a_1\psi_{j_1} \otimes \psi_{j_1'}$, $A_2 = a_2\phi_{k_1} \otimes \phi_{k_1'}$, $B_1 = b_1\psi_{j_2} \otimes \psi_{j_2'}$, $B_2 = Id_2$:

$$tr\left\{(A_1\tilde{\otimes}A_2)\widetilde{\bigotimes}(B_1\tilde{\otimes}B_2)\Sigma_C\right\} = a_1a_2b_1\sum_{k'=1}^{\infty}\beta_{j_1,k_1,j_1',k_1',j_2,k',j_2',k'}.$$

(Case 6) $A_1 = Id_1$, $A_2 = a_2\phi_{k_1} \otimes \phi_{k_1'}$, $B_1 = Id_1$, $B_2 = b_2\phi_{k_2} \otimes \phi_{k_2'}$:

$$tr\left\{(A_1\tilde{\otimes}A_2)\widetilde{\bigotimes}(B_1\tilde{\otimes}B_2)\Sigma_C\right\} = a_2b_2\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\beta_{i,k_1,i,k_1',k,k_2,k,k_2'}.$$

(Case 7) $A_1 = Id_1$, $A_2 = a_2\phi_{k_1} \otimes \phi_{k_1'}$, $B_1 = b_1\psi_{j_2} \otimes \psi_{j_2'}$, $B_2 = Id_2$:

$$tr\left\{(A_1\tilde{\otimes}A_2)\widetilde{\bigotimes}(B_1\tilde{\otimes}B_2)\Sigma_C\right\} = a_2b_1\sum_{i=1}^{\infty}\sum_{k'=1}^{\infty}\beta_{i,k_1,i,k_1',j_2,k',j_2',k'}.$$

(Case 8) $A_1 = a_1\psi_{j_1} \otimes \psi_{j_1'}$, $A_2 = Id_2$, $B_1 = Id_1$, $B_2 = b_2\phi_{k_2} \otimes \phi_{k_2'}$:

$$tr\left\{(A_1\tilde{\otimes}A_2)\widetilde{\bigotimes}(B_1\tilde{\otimes}B_2)\Sigma_C\right\} = a_1b_2\sum_{i'=1}^{\infty}\sum_{k=1}^{\infty}\beta_{j_1,i',j_1',i',k,k_2,k,k_2'}.$$

(Case 9) $A_1 = a_1\psi_{j_1} \otimes \psi_{j_1'}$, $A_2 = Id_2$, $B_1 = b_1\psi_{j_2} \otimes \psi_{j_2'}$, $B_2 = Id_2$:

$$tr\left\{(A_1\tilde{\otimes}A_2)\widetilde{\bigotimes}(B_1\tilde{\otimes}B_2)\Sigma_C\right\} = a_1b_1\sum_{i'=1}^{\infty}\sum_{k'=1}^{\infty}\beta_{j_1,i',j_1',i',j_2,k',j_2',k'}.$$

In the above, a_1, a_2, b_1, and b_2 are scalar constants. Using the above, all the terms in Θ can be obtained from straightforward but tedious calculations.

To illustrate the calculation of $\Theta(j,k,j',k',l,m,l',m')$, the term in Θ corresponding to the asymptotic covariance of $T_n(j,k,j',k')$ and $T_n(l,m,l',m')$, we consider as an example the case where $j \neq j'$, $k \neq k'$, $l \neq l'$, and $m \neq m'$. Here,

$$\begin{aligned}
&\Theta(j,k,j',k',l,m,l',m') \\
\overset{by\ Thm.\ 1\ (i)}{=}\ &\mathrm{E}\left(tr\left[\{(\psi_j\otimes\psi_{j'})\tilde{\otimes}(\phi_k\otimes\phi_{k'})\}\mathcal{Z}\right]tr\left[\{(\psi_l\otimes\psi_{l'})\tilde{\otimes}(\phi_m\otimes\phi_{m'})\}\mathcal{Z}\right]\right) \\
\overset{by\ Eq.\ (2)}{=}\ &tr\left[\{(\psi_j\otimes\psi_{j'})\tilde{\otimes}(\phi_k\otimes\phi_{k'})\}\widetilde{\bigotimes}\{(\psi_l\otimes\psi_{l'})\tilde{\otimes}(\phi_m\otimes\phi_{m'})\}\Sigma_C\right] \\
\overset{by\ Case\ 1}{=}\ &\beta_{j,k,j',k',l,m,l',m'} = \mathrm{E}(\chi_{jk}\chi_{j'k'}\chi_{lm}\chi_{l'm'}),
\end{aligned}$$

where we have used $A_1 = \psi_j \otimes \psi_{j'}$, $A_2 = \phi_k \otimes \phi_{k'}$, $B_1 = \psi_l \otimes \psi_{l'}$, and $B_2 = \phi_m \otimes \phi_{m'}$.

Proof of Lemma 4. Let $X_N(s,t) = \mu(s,t) + \sum_{j=1}^{N}\sum_{k=1}^{N}\chi_{jk}\psi_j(s)\phi_k(t)$, and let C_N denote the covariance structure of X_N. Thus,

$$C_N(s,t;u,v) = \sum_{j=1}^{N}\sum_{j'=1}^{N}\sum_{k=1}^{N}\sum_{k'=1}^{N}\mathrm{cov}(\chi_{jk},\chi_{j'k'})\psi_j(s)\psi_{j'}(u)\phi_k(t)\phi_{k'}(v).$$

It is easy to show that C_N converges to C in Hilbert–Schmidt norm. Let $C_{\mathcal{S},N} = tr_2(C_N)$, which converges to $C_{\mathcal{S}}$ because tr_2 is continuous and linear. We know that $\langle C_{\mathcal{S}}\psi_j, \psi_{j'}\rangle = 0$ for $j \neq j'$. Therefore, for any $\epsilon > 0$, we can find an N such that $|\langle C_{\mathcal{S},N}\psi_j, \psi_{j'}\rangle| < \epsilon$.

By definition,

$$\langle C_{\mathcal{S},N}\psi_j,\psi_{j'}\rangle = \int_{\mathcal{S}}\int_{\mathcal{S}}\left\{\int_{\mathcal{T}} C_N(s,t;u,t)dt\right\}\psi_j(s)\psi_{j'}(u)dsdu$$

$$= \int_{\mathcal{S}}\int_{\mathcal{S}}\int_{\mathcal{T}}\sum_{l=1}^{N}\sum_{l'=1}^{N}\sum_{k=1}^{N}\sum_{k'=1}^{N}\mathrm{cov}(\chi_{lk},\chi_{l'k'})\psi_l(s)\psi_{l'}(u)\phi_k(t)\phi_{k'}(t)\psi_j(s)\psi_{j'}(u)dtdsdu$$

$$= \sum_{k=1}^{N}\mathrm{cov}(\chi_{jk},\chi_{j'k})$$

Therefore, $\lim_N \sum_{k=1}^{N}\mathrm{cov}(\chi_{jk},\chi_{j'k}) = 0$, i.e., $\sum_{k=1}^{\infty}\mathrm{cov}(\chi_{jk},\chi_{j'k}) = 0$ for $j \neq j'$.

The same argument holds for the empirical version. Analogous calculations can be done for $k \neq k'$ to show that $\sum_{j=1}^{\infty}\mathrm{cov}(\chi_{jk},\chi_{jk'}) = 0$ and $\sum_{j=1}^{\infty} T_n(j,k,j,k') = 0$.

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[*Received* 27 *December* 2017. *Editorial decision on* 20 *April* 2018]