Generating von Mises Fisher distribution on the unit sphere ($S^2$)

Sungkyu Jung

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This short note describes the procedure for generating random unit vectors following the von Mises-Fisher distribution on the sphere $S^{m-1}$ for $m = 3$.

We use the fact (from [2], [1] and [3]) that the unit 3-vector $X$ has von Mises-Fisher distribution with modal direction $(0, 0, 1)'$ and the concentration parameter $\kappa$ if and only if

$$X = ((1 - W^2)^{\frac{1}{2}}V, W)',$$

where $V$ is the uniformly distributed unit 2-vector and $W$ (independent of $V$) is a scalar random variable with range $[-1, 1]$ and density

$$f(w) = c_\kappa^{-1} \exp(\kappa w).$$

The normalizing constant is given by

$$c_\kappa = \pi^{\frac{1}{2}} (\kappa^2)^{-\frac{1}{2}} I_{\frac{1}{2}}(\kappa) = \frac{2}{\kappa} \frac{e^\kappa - e^{-\kappa}}{2},$$

where $I_\nu(\cdot)$ is the modified Bessel function of the first kind and the second equation is given by the relation $I_{\frac{1}{2}}(x) = (\frac{x}{2})^{-\frac{1}{2}} \sinh(x)$.

The distribution function of $W$ is given by

$$F_W(t) = P(W \leq t) = c_\kappa^{-1} \left[ \frac{e^{\kappa t}}{\kappa} - \frac{e^{-\kappa t}}{\kappa} \right].$$

The quantile function $Q_W(y) = F_W^{-1}(y)$ is given by

$$Q_W(y) = \frac{1}{\kappa} \log(e^{-\kappa} + \kappa c_\kappa y).$$

Now for $Y \sim U[0, 1]$, $W = Q_W(Y) \sim F_W$.

The uniformly random unit 2-vector $V$ is easily obtained by parameterizing $V = (\cos \theta, \sin \theta)$, $\theta \sim U(0, 2\pi)$. Then a random sample of von-Mises-Fisher distribution $(m = 3)$ is obtained by Eq. (1).

See Matlab program randvonMisesFisher3.m.

References
