Multivariate Analysis homework 1

Due date: Thursday January 22, 2015.

Problems 2–6 can be solved by hand.

1. Paper is manufactured in continuous sheets several feet wide. Because of the orientation of fibers within the paper, it has a different strength when measured in the direction produced by the machine than when measured across, or at right angles to, the machine direction. The dataset in `T1-2.txt` contains 41 measurements of X_l = density (grams/cubic centimeter) X_2 = strength (pounds) in the machine direction X_3 = strength (pounds) in the cross direction. Use any statistical package to answer the following.

   (a) Report summary statistics (means and variances) of the three variables, and present a scatter plot matrix.

   (b) Identify an outlier.

2. Let $\mathbf{x}' = [5, 1, 3]$, and $\mathbf{y}' = [-1, 3, 1]$.

   (a) Graph the two vectors.

   (b) Find (i) the length of $\mathbf{x}$, (ii) the angle between $\mathbf{x}$ and $\mathbf{y}$, and (iii) the projection of $\mathbf{y}$ on $\mathbf{x}$.

3. Check that
$$\mathbf{Q} = \begin{pmatrix} \frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{pmatrix}$$
is an orthogonal matrix.

4. Let
$$\mathbf{A} = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}.$$

   (a) Is $\mathbf{A}$ symmetric?

   (b) Show that $\mathbf{A}$ is positive definite.

5. Using the matrix
$$\mathbf{A} = \begin{pmatrix} 4 & 8 & 8 \\ 3 & 6 & 9 \end{pmatrix},$$

   (a) Calculate $\mathbf{A}\mathbf{A}'$ and obtain its eigenvalues and eigenvectors.

   (b) Calculate $\mathbf{A}'\mathbf{A}$ and obtain its eigenvalues and eigenvectors. Check that the nonzero eigenvalues are the same as those in part (a).

   (c) Obtain the singular value decomposition of $\mathbf{A}$.

6. You are given a random vector $\mathbf{X} = (X_1, X_2, X_3)'$ with mean vector $\boldsymbol{\mu} = (1, 2, 3)'$ and covariance matrix
$$\Sigma = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 9 \end{pmatrix}.$$

Let $\mathbf{X}_1 = (X_1, X_2)'$, and

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

Compute the following:

(a) $\mathbb{E}(\mathbf{a}'\mathbf{X}_1)$

(b) $\mathbb{E}(\mathbf{B}\mathbf{X}_1)$

(c) $\mathrm{Cov}(\mathbf{X}_1)$

(d) $\mathrm{Cov}(\mathbf{B}\mathbf{X}_1)$

The following are extra credit questions for undergraduate students. Graduate students are required to do the problems.

7. By expressing a correlation matrix R with equal correlations ρ as $R = (1-\rho)\mathbb{I} + \rho\mathbf{J}$, where $\mathbf{J}$ is a matrix of ones, find the determinant and inverse of R.

8. Suppose that the determinant of a square symmetric matrix A is zero. Is it possible that all eigenvalues of A are positive?

9. Consider an arbitrary $n \times p$ matrix $\mathbf{A}$.

   (a) Show that $\mathbf{A}'\mathbf{A}$ is a symmetric $p \times p$ matrix.

   (b) Show that $\mathbf{A}'\mathbf{A}$ is necessarily nonnegative definite.