Generating von Mises Fisher distribution on the unit sphere (S^2)

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This short note describes the procedure for generating random unit vectors following the von Mises-Fisher distribution on the sphere S^{m-1} for $m = 3$.

We use the fact (from [2], [1] and [3]) that the unit 3-vector X has von Mises-Fisher distribution with modal direction $(0, 0, 1)'$ and the concentration parameter κ if and only if

$$X = ((1 - W^2)^{\frac{1}{2}} V, W)', \tag{1}$$

V is the uniformly distributed unit 2-vector and W (independent of V) is a scalar random variable with range $[-1, 1]$ and density

$$f(w) = c_\kappa^{-1} \exp(\kappa w).$$

The normalizing constant is given by

$$c_\kappa = \pi^{\frac{1}{2}} (\frac{\kappa}{2})^{-\frac{1}{2}} I_{\frac{1}{2}}(\kappa) = \frac{2}{\kappa} \frac{e^\kappa - e^{-\kappa}}{2},$$

where $I_\nu(\cdot)$ is the modified Bessel function of the first kind and the second equation is given by the relation $I_{\frac{1}{2}}(x) = (\frac{x}{2}\pi)^{-\frac{1}{2}} \sinh(x)$.

The distribution function of W is given by

$$F_W(t) = P(W \leqslant t) = c_\kappa^{-1} \left[\frac{e^{\kappa t}}{\kappa} - \frac{e^{-\kappa t}}{\kappa} \right].$$

The quantile function $Q_W(y) = F_W^{-1}(y)$ is given by

$$Q_W(y) = \frac{1}{\kappa} \log(e^{-\kappa} + \kappa c_\kappa y).$$

Now for $Y \sim U[0, 1]$, $W \equiv Q_W(Y) \sim F_W$.

The uniformly random unit 2-vector V is easily obtained by parameterizing $V = (\cos\theta, \sin\theta)$, $\theta \sim U(0, 2\pi]$. Then a random sample of von-Mises-Fisher distribution ($m = 3$) is obtained by Eq. (1).

See Matlab program `randvonMisesFisher3.m`.

References

[1] N. I. Fisher, T. Lewis, and B. J. J. Embleton. *Statistical analysis of spherical data*. Cambridge University Press, Cambridge, 1993. Revised reprint of the 1987 original.

[2] Gary Ulrich. Computer generation of distributions on the m-sphere. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 33(2):158–163, 1984.

[3] Andrew T.A Wood. Simulation of the von mises fisher distribution. *Communications in Statistics - Simulation and Computation*, 23(1):157 – 164, 1994.